1.2a Linear difference equations

Monday, January 11, 2021 4:56 PM

Textbook: An introduction to mathematical biology by Linda J.S. Allen

Supplement: Nonlinear dynamics and chaos by Steven Strogatz

Def. 1.1 A difference equation of order k has the form

$$f(x_{t+k}, x_{t+k-1}, \dots, x_t, t) = 0, \quad t \in \mathbb{N} = \{0, 1, 2, \dots\}$$

where $x_i \in \mathbb{R}$, and f must depend on both x_{t+k} and x_t

Note: x_i are called the state variables

Def. A difference equation is autonomous if f does not explicitly depend on t and nonautonomous otherwise.

Commonly encountered form:

$$\underbrace{x_{t+k} + \sum_{i=1}^{k} a_i(x_{t+k-1}, x_{t+k-2}, \dots, x_t, t)\, x_{t+k-i} = b_t, \quad t \in \mathbb{N}, \quad a_i(x_{t+k-1}, \dots, x_t, t) \in \mathbb{R}, \quad b_t \in \mathbb{R}}_{\text{order is } k \text{ if } a_k \neq 0.}$$

Def. 1.2 A difference equation that can be written as

$$x_{t+k} + \sum_{i=1}^{k} a_i(t)\, x_{t+k-i} = b_t$$

is called linear. Otherwise, nonlinear.

If a difference equation is linear and $b_t = 0 \ \forall t \in \mathbb{N}$, then it is homogeneous. Otherwise, it is nonhomogeneous.

Def. 1.3 A system of k first-order difference equations can be written as

$$x_i(t+1) = f_i(x_1(t), \dots, x_k(t), t), \quad i = 1, \dots, k$$

If $x_i(t+1) = f_i(x_1(t), \dots, x_k(t))$ $\forall i \in \{1, \dots, k\}$, then it is autonomous

Otherwise, nonautonomous

If $x_i(t+1) = \sum_{j=1}^{k} a_{ij}(t)\, x_j(t) + b_i(t)$, $i = 1, \dots, k$, then it is linear

If in addition, $b_i(t) \equiv 0$, for $i = 1, \dots, k$, then it is homogeneous.

Def. 1.4 A solution to a difference equation is a function $x : \mathbb{N} \to \mathbb{R}$ that makes the difference equations true.

A solution to a system of difference equations is a function $x : \mathbb{N} \times \{1, \dots, k\} \to \mathbb{R}$ that makes the difference equations true.

Often, we write a vector $X(t) = (x_1(t), \dots, x_k(t))^T$ for the solution.